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SLOPE STABILITY ANALYSIS METHODS` OPTIMIZATION COMPARISON

СРАВНЕНИЕ МЕТОДОВ ОПТИМИЗАЦИИ АНАЛИЗА СТАБИЛЬНОСТИ СКЛОНОВ

Summary. *Different approaches to sensitivity analysis and ways to affect the slope were investigated.*

Key words: *Landslide, Dynamic programming, slope stability analysis.*

Аннотация. *Исследованы различные подходы к анализу чувствительности и возможности влияния на склон.*

Ключевые слова: *Оползень, Динамическое программирование Беллмана, анализ стабильности склона.*

Introduction. In the field of slope stability, sensitivity analysis is generally conducted by means of a series of calculations in which each significant parameter is varied systematically over its maximum credible range in order to determine its influence upon the safety factor [2]. If one is interested in characterizing the variation in safety when encounter minor modifications in the parameters, these incremental techniques define an approximation of the safety factor gradient. In this case, for discrete problems (i.e., slices) the sensitivity may be calculated in a simpler and compact way by using the techniques that have been developed in the area of non-linear optimization [3]. When dealing with continuous problems as those linked to the variational approach of slope stability analysis, the formulation put forth by Castillo et al. [4] can be used.

In any event, regardless of how the sensitivity analysis is done, when instability occurs, a sensitivity analysis allows to know which qualitative or quantitative actions are more appropriate to stabilize the given slope. Therefore, the sensitivity analysis is a useful tool able to provide a sound assessment for the selection of the slope stabilization method. Our main objective in this article is to analyze the use of this sensitivity analysis tool.

It is routinely performed by two-dimensional (2D) limit equilibrium methods. For rock slopes, conventional methods developed for soil slopes, e.g. Bishop's simplified method recommended by Hoek and Bray [1], are often adopted. However, other methods, e.g. Sarma [2] which employs slices with inclined interfaces to simulate structural discontinuities, have been highly commended [3]. Unfortunately, most of these analyses are limited to two dimensions, which cannot properly model the true three-dimensional (3D) characteristics of a landslide. A practical 3D slope stability analysis method and the related computer programs are therefore urgently required [4,5].There are a large number of publications that deal with 3D slope stability analysis. In general, these methods can be classified into two categories.

The limit equilibrium approach. Duncan [6] reviewed the main aspects of 24 publications dealing with limit equilibrium approaches. The failure mass is divided into a number of columns with vertical interfaces and the conditions for static equilibrium are used to find the factor of safety. Hungr [7], Hungr et al. [8], Chen and Chameau [9] and Lam and Fredlund [10] extended Bishop's simplified, Spencer's and Morgenstern and Price's methods from two to three dimensions, respectively.

Both when limit equilibrium methods are used, and when the kinematic approach of limit analysis is applied, if the safety factor is defined as the ratio of the shear stress of the soil to the shear stress at failure, slope stability is generally evaluated as a ratio:

$$F = \frac{S}{D} = \frac{\int_a^b G(x,y(x),y\prime(x),y_G(x),y'_G(x),u(x,y);p)dx}{\int_a^b Q(x,y(x),y\prime(x),y_G(x),y'_G(x),u(x,y);p)dx} \quad (1)$$

For the two-dimensional collapse mechanism defined in Fig. 1, a and b are the x-coordinates of the sliding line end points, $y_G(x)$y is the ground profile (ordinate at point *x*), *y(x)* is the ordinate of the sliding line at point *x*, and $y'(x)$ is its first derivative. The $u(x,\ y)$ function defines the distribution of the soil water pressure.

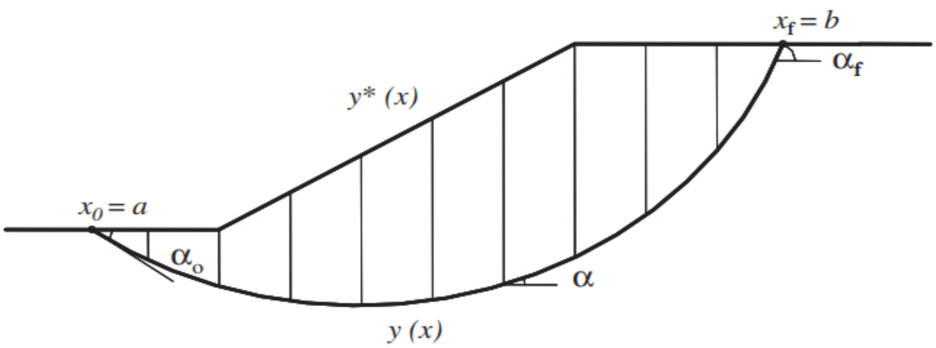

Fig. 1. Collapse mechanism in the limit equilibrium approach

Finally, vector p groups all the parameters together. In principle, it could be a vectoral field (if, for example, the spatial variation of the strength parameters is taken into account), though what usually happens is that it contains only a discrete number of parameters which are constant throughout the entire domain. G and Q are two functionals that define the actions over the system. When limit equilibrium methods are used, actions are identified with forces or moments. On the other hand, if the kinematic approach of limit analysis is used, the actions may be identified with the internal dissipation of energy, and the external work. This procedure is referred to as the ''kinematical approach" in this paper.

The slope stability problem can be stated as the minimization of the ''safety functional" (as Baker and Garber [5], termed the quotient F of Eq. (1)) to find the safety factor:

$$F = Min_{y(x),\delta(x)}\{F[y(x),\delta(x)]\} \qquad (2)$$

where $\delta(x)$ represents the distribution of stresses along $y(x)$. To minimize the quotient functional, the Petrov method can be used. Petrov [6] showed that stationary ''points" (functions), of a ratio can be obtained by extremizing an auxiliary functional $R = R(F_S) = G - F_S Q$, where F_S is the unknown minimum value of the ratio G/Q, which can be evaluated from the constraint $R(F_S) = 0$. The minimization can also be done by an iterative method. This is the case of Baker [7], when he applies the dynamic programming algorithm iteratively, i.e., assuming a value for F_S, and establishes the critical slip surface $y_{CR}(x|F_S)$, by minimizing $R\,[y(x)|F_S]$, obtaining a new estimate of F_S by applying the Spencer's procedure to this critical slip surface, and repeating the process until the assumed and resulting values of F_S are equal. Although the two preceding methods have proved its efficiency, for the approach proposed in this paper is of interest to minimize the quotient functional directly:

$$\frac{\partial F}{\partial p_i} = \frac{\partial(S/D)}{\partial p_i} = \frac{\frac{\partial S}{\partial p_i} - S\frac{\partial D}{\partial p_i}}{D^2} = \frac{1}{D}\left(\frac{\partial S}{\partial p_i} - F_S\frac{\partial D}{\partial p_i}\right) \qquad (3)$$

Then, standard optimization packages, as GAMS [8] for example, can be used. This allows to convert F_S into one more variable that has to satisfy its definition (Eq. (1)). Therefore, it is not needed to worry about the method of solution, because the method implemented in the software package takes this into account as one constraint. As a result, in keeping with the proposal of Castillo et al. [9], the components of the vector of sensitivities s of the objective function with respect to p can be defined as:

$$s_i = \frac{\partial F}{\partial p_i} = \frac{\frac{1}{D}\int_a^b \left(\frac{\partial G}{\partial p_i} - F\frac{\partial Q}{\partial p_i}\right)dx}{1 - \frac{1}{D}\int_a^b \left(\frac{\partial G}{\partial F} - F\frac{\partial Q}{\partial F}\right)dx} \qquad (4)$$

where for simplicity, the arguments of the functionals have been omitted. The safety factor local sensitivities are defined as the partial derivatives of the safety factor with respect to the parameter being studied. The partial derivatives are calculated at the optimum value. Thus, these sensitivities provide only a linear approximation in a neighborhood of the optimal point, and they only indicate the direction of the action to be taken. Since small property increments δp_i would produce significant changes in the critical slip surface, the simultaneous variation of all the variables and functions involved is taken into account, included the slip surface.

When idealized examples are analyzed, the slope stability can be studied analytically, and the sensitivity of the solution can be characterized on the basis of the parameters using Eq. (4). In many practical applications, however, these analytical computations cannot be carried out. In these cases, the S/D ratio is generally discretized by means of slices (limit equilibrium method) or blocks (kinematical approach). When these approaches are adopted, the problem is usually solved by using numerical tools [10,11]. The simplest way of approaching s consists on the application of a finite difference scheme:

$$s_i = \frac{\delta F}{\delta p_i} \approx \frac{F^+ - F^-}{2\delta p_i} \qquad (5)$$

where F^+ defines the value of F when the ith component of p is increased by δp_i, with F^- having a similar definition but with $p_i$decreasing. If vector p contains n_p components, it would be required to solve $2 \times n_p$ minimization problems, in addition to the fundamental problem for determining F. For this reason, it is more efficient to introduce the following discretization of Eq. (4) into the numerical solver:

$$s_i = \frac{\partial F}{\partial p_i} = \frac{\frac{1}{D}\sum_{k=1}^{n}(\frac{\partial G}{\partial p_i} - F\frac{\partial Q}{\partial p_i})_k \Delta x_k}{1 - \frac{1}{D}\sum_{k=1}^{n}(\frac{\partial G}{\partial F} - F\frac{\partial Q}{\partial F})_k \Delta x_k} \quad (6)$$

where n is the number of slices/blocks, and Δx_k is the horizontal width of the kth slice/block. To evaluate Eq. (6) it is necessary to have previously obtained the values of the partial derivatives

$\frac{\partial G}{\partial p_i}$, $\frac{\partial Q}{\partial p_i}$, $\frac{\partial G}{\partial F}$, and $\frac{\partial Q}{\partial F}$. To this end, the use of a symbolic programming code for obtaining closed-form expressions of these functions makes this task easier.

Once s is computed, the increase δF undergone by the safety factor after introducing a perturbation of value δp in the parameters can be estimated as:

$$\partial F = s \cdot \partial p = \sum_{i=1}^{n_p} s_i \cdot \partial p_i \quad (7)$$

This ∂p action will entail an associated cost C, which may generally be a non-linear function of the variation in the parameters:

$$C = C(\partial p) = \sum_{i=1}^{n_p} C_i(\partial p_i) \quad (8)$$

Once the cost functions C_i have been defined, the stabilization problem can be dealt with as if it were a minimization problem:

$$\min_{\partial p} C \ \ subject\ to\ \partial F = \partial F^* \quad (9)$$

i.e., what is looked for is the minimum cost required for a given improvement ∂F^* in the safety factor. Close to failure, it is usually needed to make a quick decision to prevent the collapse evolution. Then, moving from a safety factor 1.00 to a safety factor of 1.05 can be sufficient provided that the soil parameters are derived from a well-documented back analysis [12]. In this case, the linear approach (Eq. (7)) can be valid. However, if large changes are done, perhaps additional calculations must replace the linear approach to evaluate the final safety of the slope. Note that since sensitivities are partial derivatives at the optimal point, they indicate which direction to follow. These directions obviously change when one move further from the optimal point, and recalculation is then necessary.

It should be bear in mind that if the linearization of ∂F is not adopted, the constraint of Eq. (9) will entail the implicit resolution of the optimization problem related to the computation of a safety factor F equal to $F^* + \delta F^*$. This causes a significant increase of the computing time, including pre and post-processing. It will become even more evident if the stabilization project is formulated like a decision-making process (see, for example, [13]), and it will include cost/benefit analyses that will generally involve risk assessment and the probabilistic analysis of collapse and the corresponding cost (see, for example, [14]). Then, a huge number of resolutions of Eq. (9) will be needed. Although these types of studies are not within the scope of this article, the methodology put forth here can indeed be used in such cases.

However, only limited reports on the application of these methods have been documented. Stark and Eid [11] reviewed three commercially available computer programs in their attempts to analyze several landslide case histories and concluded that ''the factor of safety is poorly estimated by using commercially available software because of limitations in describing geometry, material properties and/or the analytical methods''. In general, the methods of columns with vertical interfaces suffer the following limitations:

- A large number of assumptions have to be introduced to render the problem statically determinate. Lam and Fredlund [10] balanced the number of equations that can be established from physical and mechanical requirements to the number of unknowns involved in these equations. They found that, for a failure mass divided into n rows and m columns (refer to Fig. 1), a total of 8mn assumptions are required.

- The method is further hampered by complicated 3D vector analysis that generally involves a set of nonlinear simultaneous equations. Iteration is necessary to obtain a solution unless further simplifications are introduced.
- Since the method as applied to three dimensions is in its infancy, no attempt has yet been published to find a critical 3D slip surface of a generalized shape.

The upper bound approaches. The basic principles of the upper bound theory of plasticity as applied to 2D geomechanical problems are well documented [12]. Publications dealing with this subject in three dimensions is also available [13,14]. Most of the work is based on analytical approaches in which the failure mass is divided into several blocks with simplified slip surface shapes such as straight or logarithmic lines. The often complex geometry of the surface of the slope is usually simplified to a plane described by two straight lines. The material is assumed to be homogeneous and ground water conditions are either ignored or over-simplified. These simplifications have limited the application of these methods to practical problems. Recently, Donald and Chen [15] proposed a 2D slope stability analysis method that is based on the upper bound theorem but arrives at a solution numerically. The failure mass is divided into slices with inclined interfaces. They demonstrated that this method is equivalent to Sarma's method of non-vertical slices and therefore is particularly applicable to rock slopes. However, unlike Sarma's original work that employed force equilibrium, they started the calculation by establishing a compatible velocity field and obtained the factor of safety by the energy-work balance equation. The subsequent automatic search for the critical failure modelled to success in finding accurate solutions for a number of closed-form solutions provided by Sokolovski [16]. The method described in this paper is an extension of Donald and Chen's 2D approach. The failure mass is divided into a number of prisms with inclined interfaces. It uses the upper-bound theory and therefore avoids introducing a large number of assumptions. In three dimensions, the solution for the factor of safety still remains a scalar manipulation of energy-force balance without the need for complicated non-linear 3D force equilibrium equations. Optimization routines are followed to find the critical failure mode

The upper bound method. The statement of the upper bound theorem, as it applies to soil mechanics, is described in Chen [12]. Its application to slope stability analysis is discussed by Donald and Chen [15].

For a slope that is at limit state, the material within the sliding surface, represented as Ω^*, is assumed to be plastic everywhere and therefore at yield. Under these conditions, the upper bound theorem states that among all possible external loads applied to a kinematically admissible plastic zone Ω^*, the external load T that brings about failure on a failure mode Ω, can be approached by minimizing T^* as determined from the following work-energy balance equation.

$$\int_{\Omega^*}^{\cdot} \sigma_{ij}^* \varepsilon_{ij}^* \, dv + \int_{T^*}^{\cdot} dD_S^* = WV^* + T^*V^* \qquad (10)$$

where V^* is the rate of plastic displacement, generally referred to as the plastic velocity. W is the body force corresponding to the plastic zone. The left-hand side of Eq. (1) represents the rate of internal energy dissipation within the failure mass and along the slip surface.

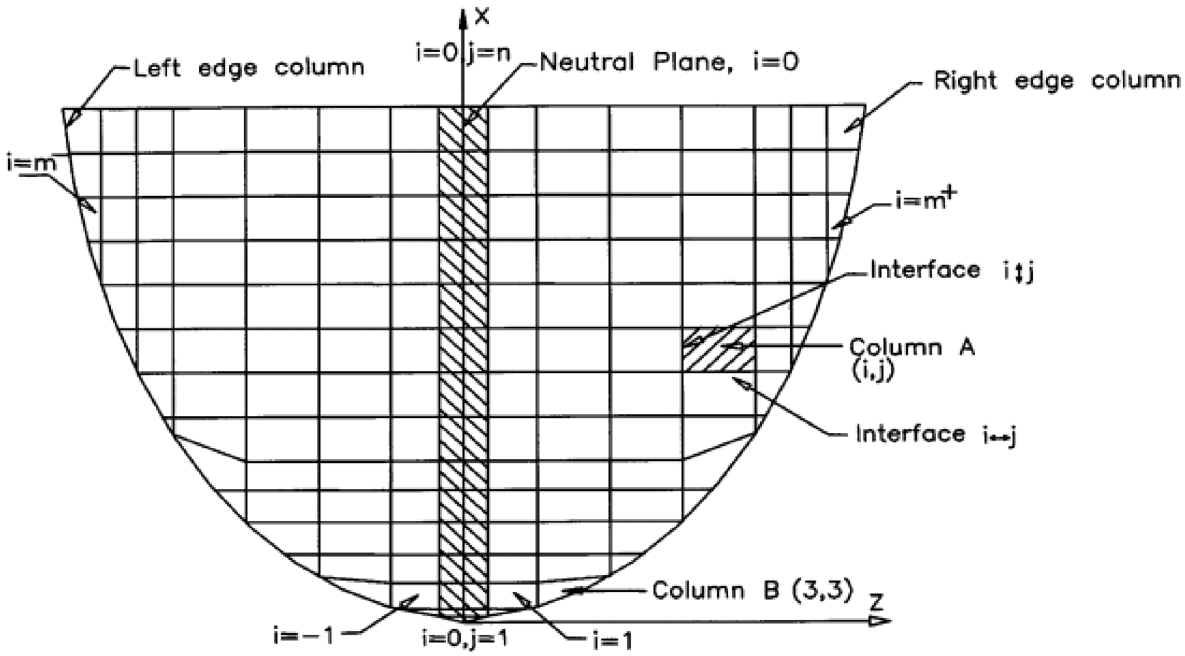

Fig. 2. 3D slope view

The 3D energy approach described here in approximates the failure mass by a series of prisms having rectangular inclined side faces (Fig. 2). For this form of discretization, Eq. (10) may be approximated in the form of a summation.

$$\sum D^*_{i\leftrightarrow J} + \sum D^*_{i\updownarrow j} + \sum D^*_{ij} = WV^* + T^*V^* \qquad (11)$$

where the symbol ↕ is used to represent the interfaces between two adjacent columns and ↔, between two adjacent rows of prisms (refer to Fig. 2). The three terms in the left-hand side of the equation approximate the energy dissipation on the row-to-row and column-to-column interfaces and on the slip surface, respectively. For a soil or rock slope that is subjected to an external load T^0, the upper bound theorem states that the loading factor η, defined as

$$\eta = \frac{T^0 - T^*}{T^0} \qquad (12)$$

should approach its minimum in order to bring the structure to failure. Other alternatives include the coefficient of critical horizontal acceleration of the body force applied on the failure mass, as suggested by Sarma [2] and discussed by Donald and Chen [15]. The main advantage of using these approaches is that η can be determined in a straightforward way from Eq. (11) without the need for iteration.

The stability of a slope is generally assessed by determining the factor (of safety), F, by which the available shear strength parameters c' and ϕ' need to be reduced to bring the structure to a limit state of equilibrium. The reduced parameters c'_e and ϕ'_e can therefore be defined by

$$c'_e = c'/F \tag{13}$$

$$\tan\phi'_e = \tan\phi'/F \tag{14}$$

The upper bound method therefore requires that the minimum value of F related to a critical failure mechanism and determined from Eq. (15) be found.

$$\sum D^*_{i\leftrightarrow J,e} + \sum D^*_{i\updownarrow j,e} + \sum D^*_{ij,e} = WV^* + T^0V^* \tag{15}$$

The three terms with subscript 'e' on the left-hand side of Eq. (15) are determined on the basis of the reduced strength parameters defined by Eqs. (13) and (14). For the remainder of this paper, the subscript 'e' is attached to any variable that has been calculated using these reduced strength parameters.

Conclusion. An effective solution procedure for the determination of the critical slip surface and its associated factor of safety has been presented. The procedure may handle any slope geometry, layering, external loads, and pore-pressure distributions. The critical slip surface is not restricted to be of any shape. The computer program SSDP couples dynamic programming minimization with the Spencer method for slope stability analysis. A similar approach may, however, be utilized for any other slope stability method which satisfies all equilibrium conditions and is valid for slip surfaces of arbitrary shape.

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